# Unexpected correspondence between noise-induced and master-slave complete synchronizations

Jian Gao,[1] Huaping Lü,[1,2] Daihai He,[3,]* Lewi Stone,[4] and Zhigang Zheng[1]
[1]*Department of Physics, Beijing Normal University, Beijing 100875, China*
[2]*Department of Physics, Xuzhou Normal University, Xuzhou 221009, China*
[3]*Department of Mathematics and Statistics, McMaster University, Hamilton, Canada L8S4K1*
[4]*Department of Zoology, Tel Aviv University, Ramat Aviv 69978, Israel*
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A comparison between noise-induced synchronization and master-slave (Pecora-Carroll) synchronization is investigated in this paper. We find an interesting correspondence between the effective driving variables of these two kinds of synchronizations in three-dimensional chaotic systems, when the systems have nonlinear terms in more than one equation. A study of the Lorenz model, the Hindmarsh-Rose neuron model, and the Hastings-Powell foodweb model is given to support this claim. It is a somewhat surprising result since these two kinds of synchronizations arise through different mechanisms. We also examine an exceptional case, where the nonlinear term of the system appears in a single equation, as in the Pikovsky-Rabinovich circuit model, and explain why the correspondence fails.

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## I. INTRODUCTION

Recently, there has been a growing interest in noise-induced synchronization [1–4]. Generally speaking, we are trained to think of noise as a disturbance that acts to destroy the order of a system or enhances chaos in a system. However, since the paper entitled *"noise-induced order"* [1] was published, investigators have begun exploring other active roles of noise. Moderate noise may enhance the directed motion of particles [2]. It was found that when the intensity of a common noise forcing exceeds a critical value, two coupled nonidentical oscillators are able to achieve phase synchronization [3] while two uncoupled identical systems are able to achieve complete synchronization (CS) [4,5]. This phenomenon is referred to as noise-induced synchronization. Another well-known method of synchronization is the master-slave method presented by Pecora and Carroll in 1990 [6,7]. Two chaotic systems can be synchronized if (i) some dynamical variables (driving variables) are used to link two systems and (ii) subsystems, excluding the driving variables, possess only negative Lyapunov exponents (LE) or more exactly, negative maximum conditional Lyapunov exponent (MCLE) [6,7] of subsystems. Though MCLE is a necessary condition but not sufficient in coupling case [8], MCLE is valid in master-slave driving scheme [9]. According to this method, given a chaotic system, whether or not CS can be realized depends on the choice of driving variables, or equivalently, the choice of subsystems. Generally, master-slave synchronization can always be realized by properly choosing driving variables [9]. Sometimes there is more than one effective driving variable, and the variables will have different effectiveness. Usually the degree of effectiveness is measured in terms of the degree of stability of the resulting CS and/or the average time spent by the master and slave systems in reaching CS (below certain tolerance) from random chosen initial states. The effectiveness can be gauged by MCLE of the subsystem. A negative MCLE corresponds to successful CS in the master-slave method and a larger absolute value corresponds to higher effectiveness. However, Gaussian white noise-induced CS cannot be realized in some systems such as Rössler oscillators [3]. This is a major difference between noise-induced CS and the master-slave scheme, where for the latter there is always at least one effective variable (to our knowledge, there must be one effective variable or a proper active-passive decomposition [9]).

In this paper we compare noise-induced CS with master-slave CS and find an unexpected correspondence between effective variables of these two kinds of CSs. In the case of master-slave scenario the onset of synchronization is determined by the conditional stability of the unperturbed attractor in the original system. While in the case of noise-induced CS, as clearly illustrated in many examples [3,4], the onset of synchronization often depends on how much and in what way the external noise perturbs the original attractor. One may expect generally that the latter should not be connected with the behavior of master-slave systems. Usually, noise-induced CS requires that the noise intensity reaches or exceeds the size of the original attractor of the system [4]. In this paper we use three different models to show there really is a correspondence. The effective variable in master-slave CS proves to be the effective variable in noise-induced CS. This is true provided that the nonlinear terms of the model appear in more than one equation of the original system, in other words, the nonlinearity cannot be eliminated simply by erasing one equation. However, if this is not the case the correspondence between master-slave and noise-induced synchronizations fails.

## II. LORENZ MODEL

First, we study the Lorenz system [10], with the following equations of motion:

$$\dot{x} = \sigma(y - x),$$

*Author to whom correspondence should be addressed.

 

$$\dot{y} = -xz + rx - y, \tag{1}$$

$$\dot{z} = xy - bz,$$

where we choose the parameters in the chaotic regime: $\sigma = 10$, $b = 8/3$, and $r = 28$. We construct the response system by letting the slave (response) system be a duplicate of a pair of the vector fields for $(x,y)$, $(y,z)$, or $(z,x)$. We examine all possible master-slave systems and set $x,y,z$ as the driving variables, respectively; as is usual, the slave subsystems are replicas of $(y,z)$, $(x,z)$, $(x,y)$ correspondingly. The LEs and subsystem's CLEs for the Lorenz model are given in the following matrix which we term the LE-CLE matrix. The first row of the matrix gives the LE of the original system, the second to fourth rows contain the subsystem's CLEs for $x \to (y,z)$, $y \to (x,z)$, and $z \to (x,y)$, respectively:

$$\begin{bmatrix} 1.3071 & 0.0000 & -21.0237 \\ 0.0000 & -2.4176 & -2.5470 \\ -4.7214 & 0.0000 & -13.6642 \\ -0.2264 & -16.0787 & 0.0000 \end{bmatrix}. \tag{2}$$

As there is a positive LE (LE$=1.3701$) in the first row, it is clear that the original system is chaotic. All Lyapunov exponents in this paper are base-2. Note that for a flow in a time continuous autonomous system, one LE must be zero [11,12], as seen in the first row. Each of the remaining rows (i.e., rows 2–4) contains the CLEs of the respective subsystem, and the MCLEs are found by seeking the largest nonzero element in each row. The absolute value of the MCLE (assuming the MCLE is negative) characterizes the effectiveness of CS state.

As the most negative of all MCLEs is found in the third row $y \to (x,z)$ (MCLE$=-4.7214$), $y$ is the most effective variable for master-slave CS; this can be shown by calculating the average transient time from randomly chosen initial states to synchronization state.

We want to emphasize here the correspondence, where $y$ for the Lorenz system is also known to be the most effective driving variable for noise-induced CS [3]. It is thus the most effective driving variable for both types of synchronizations. The second most negative MCLE is found in row 2, making $x$ the next best effective variable in both kinds of synchronizations [3,4].

## III. HINDMARSH-ROSE MODEL

Secondly, we consider the Hindmarsh-Rose (HR) [13] neuron model:

$$\dot{x} = y - ax^3 + bx^2 - z + I,$$

$$\dot{y} = c - dx^2 - y, \tag{3}$$

$$\dot{z} = r[S(x-\chi) - ,z],$$

where $a = 1.0, b = 3.0, c = 1.0, d = 5.0, S = 4.0, r = 0.006, \chi = -1.56$, and $I = 3.0$. The LE- CLE matrix of the HR neuron model is

$$\begin{bmatrix} 0.0153 & 0.0000 & -12.6777 \\ 0.0000 & -0.0089 & -1.3847 \\ -0.0793 & 0.0000 & -11.0695 \\ 0.0452 & -12.6820 & 0.0000 \end{bmatrix}. \tag{4}$$

All matrix entries in the second row $x \to (y,z)$ and the third row $y \to (x,z)$ are non-positive. Of the nonzero entries, the third row contains the most negative $(-0.0793)$, which is the most negative MCLE. Therefore, $y$ is the most effective variable in master-slave CS. Again, this corresponds to noise-induced CS where $y$ is a quite effective variable [4]. Actually the difference between MCLEs of variables $x$ and $y$ is small, this corresponds to the similarity between their critical noise intensity points for synchronization in the meaning that both are comparable to the sizes of the attractor. This is different from variable $x$ of Lorenz, which needs a very large, compared to the size of the attractor, critical noise intensity for synchronization. The HR model is similar to the Lorenz model in the sense that both have more than one effective variable in two kinds of synchronizations.

In contrast, the following Hastings-Powell foodweb model is different because it has only one effective variable for noise-induced synchronization.

## IV. HASTINGS-POWELL MODEL

We now examine common noise-induced CS in two uncoupled multitrophic chaotic ecological systems, i.e., the Hastings-Powell (1991) [14,15] tritrophic model. The model is as follows:

$$\dot{x} = rx(1-Kx) - f_1(x)y,$$

$$\dot{y} = -d_1 y - f_2(y)z + f_1(x)y \quad + D\xi(t), \tag{5}$$

$$\dot{z} = -d_2 z + f_2(y)z,$$

where $f_1 = a_1 x/(1 + b_1 x)$ and $f_2 = a_2 y/(1 + b_2 y)$. The model dynamics is chaotic for the standard parameters (Hastings-Powell 1991) $K = r = 1$, $a_1 = 5$, $a_2 = 0.1$, $b_1 = 3$, $b_2 = 2$, $d_1 = 0.4$, and $d_2 = 0.01$. The noise term $\xi(t)$ is Gaussian with $\langle \xi(t)\xi(t-\tau)\rangle = \delta(\tau)$, and $D$ denotes the noise intensity. The LE-CLE matrix of the Hasting-Powell foodweb model is

$$\begin{bmatrix} 0.0158 & 0.0000 & -0.9837 \\ 0.0000 & 0.0973 & 0.0944 \\ -0.0003 & 0.0000 & -1.1605 \\ 0.0457 & -1.0142 & 0.0000 \end{bmatrix}. \tag{6}$$

The third line for $y \to (x,z)$ is actually the only line with a zero MCLE $(-0.0003)$. So $y$ is a neutral variable for master-slave CS. This is consistent with noise-induced CS, where $y$ is the only effective variable as shown in Fig. 1, [16]. Here we plot the average synchronization error $|X_1 - X_2|$ against

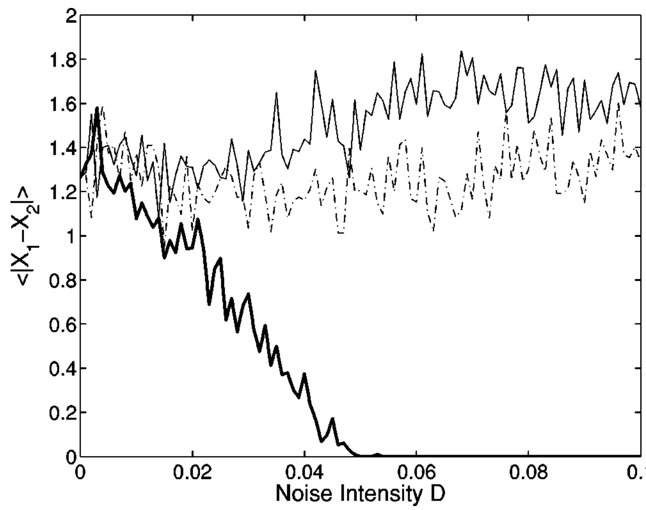


FIG. 1. The average synchronization error $|X_1-X_2|$ vs noise intensity $D$ while noise acting on $x$ (thin line), $y$ (bold line), and $z$ (dash line), respectively. The critical point for CS while noise acting on $y$ can be seen at $D_c=0.05$, above which the error vanishes. Noise-induced synchronization cannot be realized by noise acting on either $x$ or $y$.

the noise intensity $D$, where noise acts on $x$ (thin line), $y$ (bold line), and $z$ (dash line), respectively. The critical point for $y$ can be seen at $D_c=0.05$, below which there is no CS. When $D>D_c$, two uncoupled identical models driven by common noise are completely synchronized.

The above three models illustrate a correspondence between noise-induced and master-slave synchronizations. This is surprising, because the mechanism of these two kinds of synchronizations are quite different. For master-slave CS, the CLE is a known useful exponent for determining the effectiveness of synchronization. However, for noise-induced CS, no such criterion has been found in the theory. The interesting correspondence we find here suggests a criterion for diagnosing noise-induced CS. Nevertheless, the correspondence is not completely general as we proceed to show below.

## V. PIKOVSKY-RABINOVICH MODEL

An inspection of the three models discussed above reveals that each has nonlinear terms in more than one equation. We now consider a different case when the nonlinear term appears in a single equation. In this case nonlinearity can be eliminated simply by erasing the equation including the nonlinear term. In master-slave scheme, if we choose the variable, whose equation contains nonlinear term, as the driving variable, when we calculate MCLE of the subsystem, it is obvious that the Jacobian matrix of the subsystem is a constant matrix, since the subsystem contains no nonlinear term. Master-slave CS cannot be realized when the subsystem has positive MCLE, actually in this case MCLE can be given analytically since it has constant Jacobian matrix, MCLE is based on Jacobian matrix [11,12].

Now let us consider the Pikovsky-Rabinovich (PR) [17] circuit model. The equations are as follows:

$$\begin{aligned}\dot{x}&=y-\beta z,\\ \dot{y}&=-x+2\gamma y+\alpha z,\\ \dot{z}&=(x-z^3+z)/\mu,\end{aligned}\tag{7}$$

where $\beta=0.66,\alpha=0.165,\gamma=0.201$, and $\mu=0.047$. In contrast to the cases mentioned above, the PR circuit exhibits different behaviors even though it has a two-wing structure similar to the Lorenz system [17]. The LE-CLE matrix of the system is

$$\begin{bmatrix} 0.1481 & 0.0000 & -69.1866\\ 0.0000 & 0.5686 & -69.8410\\ -0.4535 & 0.0000 & -69.2213\\ 0.2147 & 0.2147 & 0.0000\end{bmatrix}.\tag{8}$$

Here $y\rightarrow(x,z)$ has the most negative MCLE in the third row, making $y$ the most effective variable in master-slave CS. Surprisingly, however, it can be shown that $z$ is the only effective variable for generating noise-induced CS [4]. This interesting difference can be understood from the equations of motion, Eq. (7). It can be seen that the only nonlinear term appears in the equation of $\dot{z}$, thus the Jacobian matrix of the system includes single variable of $z$ and at the same time only in the third row. The mechanism of noise-induced CS is actually changing the distribution of $z$ values on the disturbed attractor to make the maximum LE negative [3,4]. So, it is not strange that $z$ is the most effective in noise-induced CS. However, on the other hand, the calculation of CLE of subsystem $(x,y)$, i.e., assuming $z$ as driving variable in master-slave CS, is equivalent to eliminating the third line of Jacobian matrix, unfortunately the rest part of the Jacobian matrix is constant and yields positive maximum CLE, so $z\rightarrow(x,y)$ scheme in master-slave CS fails and should be avoided, while noise-induced CS on $z$ can be realized effectively. Furthermore, this may be a reason that $y$ is effective in master-slave CS rather than in noise-induced CS. $y$ is the overlapping variable of the two unstable subsystems $(y,z)$ and $(x,y)$. Very weak noise applied on $y$ will make the system blow out. There must be some overlapping variables when there is more than one unstable subsystem in a three-dimensional system. For dimension greater than three, e.g., four-dimensional kinetic rate neural spike model [18], the effective variables in these two kinds of CSs are reversed. This may be due to more than one unstable subsystems. Though in this paper only three-dimensional systems are considered, in higher-dimensional case, an analysis on the overlapping variables of the unstable subsystems may be helpful.

## VI. DISCUSSION AND CONCLUSION

In this paper we have compared two important synchronizations, noise-induced CS and master-slave CS. Our numerical results show an interesting correspondence between effective variables in these two kinds of CSs in chaotic systems (described by three equations) as long as there are nonlinear terms in more than one equation. However, for models in which the nonlinear term appears in only a single equation, this correspondence fails. The mechanism can be heuristically understood. The latter case has a virtue, where the effective variable for noise-induced synchronization is easily determined. The variable that is effective in noise-induced CS may not be effective in master-slave CS. This is because that when this variable is used as a driving variable in master-slave scheme, a replica of the rest subsystem will have a constant Jacobian matrix and yield positive MCLE. On the other hand, the effective variable in master-slave CS may not be effective in noise-induced CS in this case. The reason is that the variable is an overlapping variable of two unstable subsystems in the meaning of master-slave scheme, which is more unstable.

In this Brief Report we link noise-induced CS to the well-known master-slave CS. Since chaos synchronization, especially noise-induced CS, has potential important applications in circuit and biological (neuron and ecology) fields, we hope our work can inspire relative researches in these fields.

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